EFFECTIVE LOW ENERGY THEORIES AND QCD DIRAC SPECTRA

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We analyze the smallest Dirac eigenvalues by formulating an effective theory for the QCD Dirac spectrum. We find that in a domain where the kinetic term of the effective theory can be ignored, the Dirac eigenvalues are distributed according to a Random Matrix Theory with the global symmetries of the QCD partition function. The kinetic term provides information on the slope of the average spectral density of the Dirac operator. In the second half of this lecture we interpret quenched QCD Dirac spectra at nonzero chemical potential (with eigenvalues scattered in the complex plane) in terms of an effective low energy theory.

1 Introduction

There are two essentially different approaches to many-body problems. The first approach is to study them by means of Monte-Carlo simulations of the microscopic theory. The second approach is to isolate the relevant degrees of freedom and to describe them by an effective theory. Both approaches have their merits and generally their complementarity leads to a deeper understanding of the underlying phenomena. In this lecture we will focus on Quantum Chromo-Dynamics (QCD) which is the theory of the strong interactions. A great deal of effort has been devoted to Monte Carlo simulations of lattice QCD [1]. They provide a firm footing for our understanding of nonperturbative phenomena such as confinement and chiral symmetry breaking. For QCD at low energy an alternative approach is possible. Because of spontaneous breaking of chiral symmetry, QCD at low energy reduces to a theory of weakly interacting Goldstone bosons. Although this theory cannot be derived from QCD by means of an ab-initio calculation, its Lagrangian is determined uniquely by chiral symmetry and Lorentz invariance [2]. The validity of this low-energy theory is based on the presence of a mass-gap which is a highly nontrivial and nonperturbative feature of QCD.

Chiral symmetry is spontaneously broken at low temperatures. Mainly through lattice simulations, it is now widely accepted that a chiral restoration transition takes place at a temperature of about 140 MeV. The order parameter of this transition is the density per unit volume of Dirac eigenvalues near zero. This is the reason why we are interested in the properties of the smallest eigenvalues of the Dirac operator [3,4]. The situation at finite baryon density and zero temperature is much less well understood. Monte-Carlo simulations are only possible if the fermion determinant is ignored. It has been shown that the so called quenched approximation fails spectacularly at nonzero chemical potential [5,6]. On general grounds it is expected that a chiral phase transition occurs at a value of the chemical potential where it becomes advantageous to create the lightest particles with nonzero

baryon number. In QCD these are the nucleons, but in the quenched approximation, they are Goldstone modes made from quarks and conjugate anti-quarks [6]. At nonzero chemical potential, the relation between the chiral order parameter and the QCD Dirac spectrum, which is now scattered in the complex plane, is much less transparent. The failure of quenching can be understood as the absence of spectral ergodicity; at finite density the ensemble averaged Dirac spectrum is completely different form the spectral averaged Dirac spectrum. We will investigate the Dirac spectrum by means of an effective theory. Our approach is similar to the one used for QCD with two colors and for QCD with adjoint quarks [7,8]. By investigating the properties of the smallest Dirac eigenvalues we hope to obtain a better understanding of this problem and other related issues.

In this lecture we wish to discuss to what extent spectra of the QCD Dirac operator can be derived from an effective low energy partition function. In the first half of this talk we formulate an effective theory for the Dirac spectrum based on a spectrum generating function that in addition to the usual quarks contains bosonic ghost quarks to properly normalize the spectral density. This trick has been widely used in the super-symmetric formulation of Random Matrix Theory [9,10,11]. We identify a domain where the results for the resolvent are given by a chiral Random Matrix Theory with the symmetries of the QCD partition function. We find that our results are in agreement with recent lattice simulations. In the second half of this lecture we discuss quenched QCD Dirac spectra at nonzero chemical potential in terms of an effective theory. This lecture is based on several recent articles [12,13,14,15,16,17] and additional background material can be found in several recent reviews [18,19,20].

2 The QCD partition function

The QCD partition function for N_f quarks with mass m, temperature T and quark chemical potential μ is given by

$$Z(m,\mu,T) = \mathrm{Tr}\, e^{-\frac{H_{\mathrm{QCD}}-\mu N}{T}}, \tag{1}$$

where H_{QCD} is the Hamiltonian of QCD and N is the quark number operator. The trace is over all states of the theory. This partition function can be rewritten as a Euclidean Feynman path integral

$$Z(m,\mu,T) = \langle \det{}^{N_f}(D+m)\rangle_{\mathrm{YM}}, \tag{2}$$

where the average of the fermion determinant is over the Euclidean Yang-Mills action and the Dirac operator is denoted by D. In this lecture we focus on the basic symmetry properties of D and its explicit representation is not necessary. For simplicity we take all N_f quark masses equal to m.

2.1 The eigenvalues of the Dirac operator

The main subject of this talk are the properties of the smallest eigenvalues of the QCD Dirac operator. In a chiral representation of the gamma-matrices, the

Euclidean Dirac operator has the block structure

$$D = \begin{pmatrix} 0 & id + \mu \\ id^\dagger + \mu & 0 \end{pmatrix}, \tag{3}$$

where d is the covariant derivative of the color group. For $\mu \neq 0$ the Dirac operator does not have any hermiticity properties and the eigenvalues will be scattered in the complex plane. We will return to this case in the second half of this talk. For $\mu = 0$ the Dirac operator is anti-Hermitian, $D^\dagger = -D$ and has purely imaginary eigenvalues. The nonzero eigenvalues occur in pairs $\pm i\lambda_k$, whereas zero eigenvalues related to the topology of the gauge fields remain unpaired. For simplicity we will restrict the discussion in this lecture to gauge fields with a trivial topology where all eigenvalues occur as pairs. For example, our gauge fields could be composed of fields of an equal number of instantons and anti-instantons [4]. In terms of its eigenvalues, the Euclidean partition function can be written as

$$Z(m, \mu, T) = \left\langle \prod_k (i\lambda_k + m)^{N_f} \right\rangle_{\rm YM}. \tag{4}$$

The order parameter of the chiral phase transition is the chiral condensate

$$\begin{aligned} \Sigma &= \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V N_f} \partial_m \log Z \\ &= \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \sum_k \left\langle \frac{1}{m + i\lambda_k} \right\rangle_{\rm QCD} = \lim_{m \to 0} \lim_{V \to \infty} \frac{1}{V} \sum_{\lambda_k > 0} \left\langle \frac{2m}{m^2 + \lambda_k^2} \right\rangle_{\rm QCD}, \end{aligned} \tag{5}$$

where the average with label QCD includes both the fermion determinant and the Yang-Mills action. At finite volume, the chiral condensate is zero for $m = 0$. Only if the thermodynamic limit is taken before the chiral limit ($m \to 0$) can the chiral condensate become nonzero. If this is the case, chiral symmetry is broken spontaneously. The situation is analogous to the magnetization in a Heisenberg model. At finite volume and zero magnetic field the magnetization is zero. Spontaneous magnetization arises if the thermodynamic limit is taken before putting the external field to zero. According to Goldstone's theorem spontaneous breaking of a continuous symmetry implies the existence of massless Goldstone bosons. Because of confinement QCD has a mass gap and, at low energy, the partition function is dominated by the Goldstone modes. The effective theory describing their interactions then follows from chiral symmetry and Lorentz invariance.

One of the questions we would like to address in this lecture is whether there is a relation between the existence of a well-defined low-energy theory and the spectrum of the Dirac operator. A second, seemingly unrelated question is the connection of this low-energy with Random Matrix Theory.

3 Symmetries of the Spectrum Generating Function at $\mu = 0$

We will study the Dirac spectrum by means of the resolvent defined by

$$\Sigma(z) = \frac{1}{V} \left\langle {\rm Tr} \frac{1}{D + z} \right\rangle_{\rm QCD}. \tag{6}$$

The mass z in the resolvent is not related to the mass m in the fermion determinant. Such mass is known in the literature as a valence quark mass. However, we will use the more appropriate name of spectral mass.

In terms of the spectral density

$$\rho(\lambda) = \langle \sum_k \delta(\lambda - \lambda_k) \rangle_{\rm QCD}, \tag{7}$$

the resolvent can be written as

$$\Sigma(z) = \frac{1}{V} \int \rho(\lambda) d\lambda \frac{1}{i\lambda + z}. \tag{8}$$

Therefore, $\Sigma(z)$ is an analytic function with a cut on the imaginary axis. This identity can be inverted by taking the discontinuity across the cut

$$\frac{\rho(\lambda)}{V} = \lim_{\epsilon \to 0} \frac{1}{2\pi} (\Sigma(i\lambda + \epsilon) - \Sigma(i\lambda - \epsilon)). \tag{9}$$

In order to obtain a generating function for $\Sigma(z)$ we use a method that has been widely used in the theory of disordered systems [9,11], namely [21,15,16]

$$Z^{\rm sp}(z, J, m) \;=\; \langle \frac{\det(D + z + J)}{\det(D + z)} {\det}^{N_f}(D + m) \rangle_{\rm YM}. \tag{10}$$

The resolvent is then given by

$$\Sigma(z) = \frac{1}{V} \partial_J Z(z, J, m) \Big|_{J=0}. \tag{11}$$

The spectrum generating function (10) contains $N_f + 1$ fermionic quarks and one bosonic quark. In addition to the chiral symmetry, this partition function also contains a super-symmetry that mixes fermionic and bosonic quarks. This can be seen by rearranging the fermion determinant as

$$\det \begin{pmatrix} m & id \\ id^\dagger & m \end{pmatrix} = \det \begin{pmatrix} id & m \\ m & id^\dagger \end{pmatrix}. \tag{12}$$

We observe that for $m = z = J = 0$ the partition function is invariant under

$$\begin{pmatrix} id & & & & & \\ & \ddots & & & & \\ & & id & & & \\ & & & id^\dagger & & \\ & & & & \ddots & \\ & & & & & id^\dagger \end{pmatrix} \to \begin{pmatrix} U & \\ & V \end{pmatrix} \begin{pmatrix} id & & & & & \\ & \ddots & & & & \\ & & id & & & \\ & & & id^\dagger & & \\ & & & & \ddots & \\ & & & & & id^\dagger \end{pmatrix} \begin{pmatrix} U^{-1} & \\ & V^{-1} \end{pmatrix}, \tag{13}$$

where U an V are $(N_f + 1|1) \times (N_f + 1|1)$ super-matrices. Mathematically, this symmetry group is known as $Gl(N_f + 1|1) \times Gl(N_f + 1|1)$. A $Gl(1)$ subgroup is broken by the anomaly. The chiral condensate $\Sigma \equiv \langle \bar\psi \psi \rangle$ is only invariant under the diagonal subgroup with $U = V$. Therefore the symmetry is broken spontaneously

to $Gl(N_f+1|1)$. As is the case in QCD, the masses of the Goldstone modes are given by the Gell-Mann-Oakes-Renner relation

$$\frac{\sqrt{2m\Sigma}}{F}, \qquad \frac{\sqrt{(m+z)\Sigma}}{F}, \qquad \frac{\sqrt{2z\Sigma}}{F}, \tag{14}$$

where F is the pion decay constant. The Goldstone modes corresponding to a fermionic and a bosonic quark are fermionic whereas all other Goldstone modes are bosonic.

4 Effective Low-Energy Theory

The manifold $Gl(N_f+1|1)$ is not Riemannian, and is therefore not suitable as a Goldstone manifold. The Goldstone manifold is given by the maximum Riemannian submanifold of the symmetric superspace $Gl(N_f+1|1)$ which will be denoted by $\hat{G}l(N_f+1|1)$ [22,15,16]. If we ignore certain complications related to the topological structure of the QCD vacuum, this partition function is given by

$$Z(m,J,z) = \int_{\hat{G}l(N+f+1|1)} dU e^{-\int d^4x \left[\frac{F^2}{4}\mathrm{Str}\,\partial_\mu U \partial_\mu U^{-1} - \frac{1}{2}\Sigma \mathrm{Str}\, M(U+U^{-1})\right]}, \tag{15}$$

where the mass matrix $M = \mathrm{diag}(m,\cdots,m,z+J,z)$.The super-matrix U is parameterized as [15,16]

$$U = \begin{pmatrix} V & \alpha \\ \beta & e^s \end{pmatrix} \equiv e^{i\sqrt{2}\Phi/F}. \tag{16}$$

Here, V is a $U(N_f+1)$-matrix, α and β are Grassmann valued vectors of length N_f+1 and s is a real number.

In order to estimate the relative importance of the two terms in the effective Lagrangian we expand the fields to second order in the pion fields $\Phi = \pi^a t_a$,

$$\mathcal{L}^{\mathrm{eff}} = \frac{1}{2}\partial_\mu \pi^a \partial_\mu \pi^a + \frac{1}{2}M_a^2 \pi_a^2, \tag{17}$$

where M_a is one of the Goldstone masses given in (14). In a box of volume L^4, the smallest nonzero momenta are of the order $p_\mu \sim 1/L$. Let us consider QCD in the chiral limit (with $m=0$). Then the regular mesons are massless whereas mesons containing one or two spectral quarks have a nonzero mass given by (14). Therefore, if [12]

$$\frac{z\Sigma}{F^2} \ll \frac{1}{L^2} \tag{18}$$

the correlation functions involving spectral quarks are dominated by contributions from the zero momentum Goldstone modes. Nonperturbatively, the partition function reduces to a group integral in this limit. This integral has been calculated analytically [15,16] resulting in the following dependence of the condensate on the spectral mass

$$\frac{\Sigma(z)}{\Sigma} = \mu_z \left[I_{N_f}(\mu_z)K_{N_f}(\mu_z) + I_{N_f+1}(\mu_z)K_{N_f-1}(\mu_z)\right] . \tag{19}$$

As already could be observed from the generating function, it depends on z only through the combination $\mu_z \equiv zV\Sigma$. This result was first obtained [12] by means of chiral Random Matrix Theory to be discussed in the next section.

QCD is not the only theory that reduces to this effective partition function. In fact, any partition function with the same pattern of chiral symmetry breaking and a mass gap reduces to the same low-energy theory. Explicit examples are given by the random flux model [23] and other models with a hopping term, disorder and chiral symmetry [24,25]. A natural question is what is the simplest theory with the same zero momentum sector as QCD. This theory is chiral Random Matrix Theory.

5 Chiral Random Matrix Theory

In the sector of topological charge ν and for N_f quarks with mass m, the chiral random matrix partition function with the global symmetries of the QCD partition function is defined by [4,26]

$$Z_\beta^\nu(\mathcal{M}) = \int DW \prod_{f=1}^{N_f} \det \begin{pmatrix} m & iW \\ iW^\dagger & m \end{pmatrix} e^{-\frac{N\beta}{4}\Sigma \mathrm{Tr}\, W^\dagger W}, \tag{20}$$

where W is a $n\times(n+|\nu|)$ matrix and $N = 2n+|\nu|$. As is the case in QCD, we assume that the equivalent of the topological charge ν does not exceed $\sqrt{N}$, so that, to a good approximation, $n = N/2$. Then the parameter Σ can be identified as the chiral condensate and N as the dimensionless volume of space time (Our units are defined such that the density of the modes $N/V = 1$). The matrix elements of W are either real ($\beta = 1$, chiral Gaussian Orthogonal Ensemble (chGOE)), complex ($\beta = 2$, chiral Gaussian Unitary Ensemble (chGUE)), or quaternion real ($\beta = 4$, chiral Gaussian Symplectic Ensemble (chGSE)). For QCD with three or more colors and quarks in the fundamental representation the matrix elements of the Dirac operator are complex and we have $\beta = 2$. The ensembles with $\beta = 1$ and $\beta = 4$ are relevant in the case of two colors and adjoint fermions, respectively. The reason for choosing a Gaussian distribution of the matrix elements is its mathematical simplicity. It can be shown that the correlations of the eigenvalues on the scale of the level spacing do not depend on the details of the probability distribution [27,28,29,30,31,32,33,34,35,36].

Together with the Wigner-Dyson ensembles and the superconducting random matrix ensembles [37] the chiral ensembles can be classified according to the Cartan classification of large symmetric spaces [22].

6 Scales in the Dirac Spectrum

For a nonzero value of the chiral condensate Σ we can identify three important scales in the Dirac spectrum. The first scale is the smallest nonzero eigenvalue of the Dirac operator given by $\lambda_{\min} = 1/\rho(0) = \pi/\Sigma V$. The second scale is the spectral mass for which the Compton wavelength of the associated Goldstone bosons is equal to the size of the box. As discussed above this scale is given by [38,39,12,13]

$$E_c = \frac{F^2}{\Sigma L^2}. \tag{21}$$

In mesoscopic physics [39] this scale is known as the Thouless energy. It is given by the inverse diffusion time of an electron through a sample of length L. In Euclidean QCD, this scale can be interpreted in terms of diffusive motion of quark in 4 Euclidean dimensions and one artificial time dimension [13]. A third scale is given by a typical hadronic mass scale. The three scales are ordered as $\lambda_{\min} \ll E_c \ll \Lambda$.

For spectral masses $z \ll E_c$ the kinetic term in the effective action can be neglected and the low-energy partition function can be reduced to a zero dimensional integral. As already mentioned, the results for $\Sigma(z)$ obtained from the spectral partition function and from chRMT coincide [15,16] in this domain.

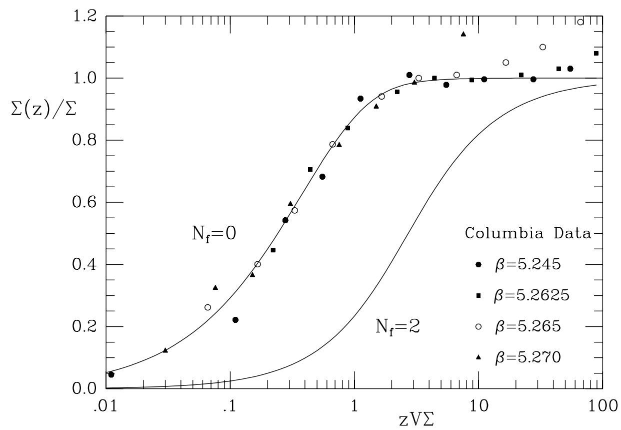

Figure 1. The spectral mass dependence of the chiral condensate $\Sigma(z)$ plotted as $\Sigma(z)/\Sigma$ versus $zV\Sigma$. The dots and squares represent lattice results by the Columbia group [40] for values of β as indicated in the label of the figure.

In the domain $E_c \ll z \ll \Lambda$ the kinetic term has to be taken into account. The slope of the Dirac spectrum at $\lambda = 0$ can be obtained from a one-loop calculation. For N_f massless flavors the result is given by [41,15,17]

$$\frac{\rho'(0)}{\rho(0)} = \frac{(N_f - 2)(N_f + \beta)}{16\pi\beta} \frac{\Sigma_0}{F^4}. \tag{22}$$

Here, β denotes the Dyson index of the Dirac operator defined before.

The domain below E_c has been investigated extensively by means of lattice QCD simulations and complete agreement with the chRMT results has been found [12,42,43,44,45,46,47,48,49, 50,51,52,53,54,55,56,57,58]. In Fig. 1 we show a comparison of the ratio $\Sigma(z)/\Sigma$ versus $zV\Sigma$ of lattice results obtained by the Columbia group [40] and eq. (19) for $N_f = 0$ and 2. The point where the lattice data depart from the chRMT result roughly coincides with the scale E_c defined in eq. (21).

7 QCD Dirac spectra at $\mu \neq 0$

The QCD partition function at $\mu \neq 0$ is given by

$$Z = \sum_\alpha e^{-\frac{E_\alpha - \mu N_\alpha}{T}}. \tag{23}$$

Below we will derive the Dirac spectrum from the property that for $T \to 0$ only states with $E_\alpha/N_\alpha < \mu$ contribute to the partition function.

For $\mu \neq 0$ the eigenvalues are scattered in the complex plane. The spectral density is then given by [59]

$$\rho(z) = \frac{1}{\pi}\partial_{z^*} G(z), \tag{24}$$

with the resolvent $G(z)$ defined as usual by

$$G(z) = \left\langle \frac{1}{V} \mathrm{Tr}\, \frac{1}{D+z} \mathrm{det}^{N_f}(D+m) \right\rangle_{\mathrm{YM}} \tag{25}$$

Writing out explicitly the real and imaginary parts, we observe that $G(z)$ is the electric field in the plane of charges located at the positions of the eigenvalues. The spectrum of the Dirac operator has been studied extensively in the context of Random Matrix Theory [6,60,61,62,63,64,65,66,67 , 68,69,70,71]. Below, we will obtain the quenched Dirac spectrum using an effective low energy. To lowest order in μ^2 our results agree with quenched Random Matrix Theory.

At $\mu \neq 0$ a new ingredient enters in the definition of the the spectrum generating function. In order to assure convergence of the bosonic integral we need an additional factor $\det(D^\dagger + z^*)$ in the denominator, and a corresponding factor in the numerator,

$$Z(z, m) = \left\langle \frac{\det(D+z+J)\det(D^\dagger + z^* + J^*)\mathrm{det}^{N_f}(D+m)}{\det(D+z)\det(D^\dagger + z^*)} \right\rangle_{\mathrm{YM}}. \tag{26}$$

The additional factor can be interpreted in terms of conjugate quarks with a baryon number opposite to that of ordinary quarks [6]. This opens the possibility of baryonic Goldstone modes consisting of a quark and a conjugate anti-quark. For simplicity we only discuss the quenched case where $N_f = 0$.

In order to construct the effective partition function we have to identify the symmetries of the spectrum generating function. To this end we rewrite the product of the determinant and the conjugate determinant as follows

$$\begin{aligned}
\det(D+z)\det(D+z^*) &= \det\begin{pmatrix} id+\mu & z \\ z & id^\dagger + \mu \end{pmatrix} \det\begin{pmatrix} -id+\mu & z^* \\ z^* & -id^\dagger + \mu \end{pmatrix} \\
&= \det\begin{pmatrix} id+\mu & 0 & z & 0 \\ 0 & id-\mu & 0 & -z^* \\ z & 0 & id^\dagger + \mu & 0 \\ 0 & -z^* & 0 & id^\dagger - \mu \end{pmatrix}
\end{aligned} \tag{27}$$

As can be observed from the sign of the chemical potential, the baryon number of the conjugate quarks is opposite to that of the usual quarks.

In this case the chiral symmetry is broken spontaneously according to

$$Gl(2|2) \times Gl(2|2) \to Gl(2|2). \tag{28}$$

This symmetry is broken explicitly by the term proportional to μ as

$$Gl(1,1|1,1) \times Gl(1,1|1,1). \tag{29}$$

This gives rise to an additional term in the effective action. In order to maintain invariance under both the left and right symmetry groups we need at least a term of second order in μ and U. One easily verifies that to this order, the only static term with the correct symmetry properties is given by

$$\int d^4x \mu^2 \mathrm{Str}\, U T_3 U^{-1} T_3, \tag{30}$$

where the symmetry breaking matrix T_3 is a diagonal matrix with nonzero matrix elements equal to 1 except on the positions corresponding to the conjugate quarks, where they are -1. For example, for $N_f = 0$ we have

$$T_3 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{31}$$

If we ignore complications related to the topology of the gauge field configurations, the static part of the effective partition function is given by

$$\int_{U \in \hat{G}l(2|2)} e^{\frac{1}{2}\Sigma V \mathrm{Str}\, M(U+U^{-1}) - \frac{1}{2} V \mu^2 F^2 \mathrm{Str}\, U T_3 U^{-1} T_3}, \tag{32}$$

where the mass matrix is given by $M = \mathrm{diag}(z+J, z^*+J^*, z, z^*)$. The integral is over the maximum Riemannian submanifold of $Gl(2|2)$. The relative coefficient of the two symmetry breaking terms is determined by the condition that the partition function should have a singularity when 2μ becomes equal to the mass of the lightest meson. The coefficient of the term $\sim \mu^2$ has to be chosen such that the effective meson mass vanishes at

$$\mu^2 = \frac{\mathrm{Re}\, z\, \Sigma}{2F^2}, \tag{33}$$

guaranteeing a nonanlyticity of the μ-dependence at this point. The value of this coefficient can be obtained more elegantly by means of a gauge principle. This construction has been performed for QCD with two colors and for QCD with adjoint fermions [7,8]. In addition to the term proportional to μ^2 we then obtain the coefficients of the terms in the effective Lagrangian that are linear in μ and the momentum.

Baryonic Goldstone modes contain one ordinary quark and one conjugate quark both with mass z resulting in a square mass of $2\mathrm{Re}\, z\, \Sigma/F^2$. For $\mu^2 < \mathrm{Re}\, z\, \Sigma/2F^2$ only the vacuum state contributes to the QCD partition function, and thus

$$Z(z, J) = e^{2\mathrm{Re}\, J\, \Sigma V}. \tag{34}$$

In the effective theory this corresponds to the saddle-point $U = 1$. For $\mu^2 > \mathrm{Re}\, z\, \Sigma/2F^2$, baryonic Goldstone modes contribute to the partition function resulting in a nonzero baryon density. The density stays finite because of the repulsive interaction between the Goldstone modes. In terms of the effective partition function this happens by the rotation of the saddle point from $U = 1$ to a nontrivial value. By making the Ansatz for the boson-boson and the fermion-fermion blocks

$$U_{\rm BB} = U_{FF} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{35}$$

one finds that $\cos\theta = \Sigma \mathrm{Re}\, z\, /2\mu^2 F^2$. A similar rotation of the saddle point has been found in the analysis of nonhermitian random matrix models [64,63,65]. We then find the following J-dependence of the partition function

$$Z(z, J) = \exp[\frac{V\Sigma^2 \mathrm{Re}\, J\, \mathrm{Re}\, z}{\mu^2 F^2}]. \tag{36}$$

The resolvent in both domains follows by differentiation with respect to the source J. For $\mathrm{Re}\, z > 2\mu^2 F^2/\Sigma$ we find from (34)

$$G(z) = \Sigma \quad \text{and} \quad \rho(z) = 0, \tag{37}$$

and for $\mathrm{Re}\, z < 2\mu^2 F^2/\Sigma$ the result for the resolvent following from (36) is given by

$$G(z) = \frac{\Sigma^2 \mathrm{Re}\, z}{2\mu^2 F^2} \quad \text{and} \quad \rho(z) = \frac{\Sigma^2}{4\mu^2 F^2}. \tag{38}$$

We observe that the resolvent is continuous at the transition point. The eigenvalues are thus located inside a strip of width $4F^2\mu^2/\Sigma$. In terms of the interpretation of the resolvent as an electric field, an eigenvalue density that does not depend on the imaginary part of z results in a constant electric field outside of the strip of eigenvalues. Ignoring finite size effects this is indeed what is observed in quenched lattice QCD simulations [5]. In the quenched approximation these results are in agreement with an explicit Random Matrix Model calculation. Therefore, to order μ^2 the Random Matrix partition function reproduces exact QCD results.

8 Conclusions

We have shown that the correlations of QCD Dirac eigenvalues at a scale well below the hadronic mass scale can be obtained analytically. They are given by an effective theory for the generating function of the resolvent for the QCD Dirac spectrum. For mass scales below $F^2/\Sigma L^2$ the kinetic term in the effective theory can be ignored resulting in eigenvalue correlations given by chRMT. Our results have been confirmed by numerous lattice QCD simulations. The same procedure can be followed for QCD at nonzero chemical potential. Our results indicate that in the quenched limit the eigenvalues are scattered in a strip with a width determined by the lightest meson of our theory. This shows that, to lowest nontrivial order in the chemical potential, chiral Random Matrix Theory provides an exact description of the QCD Dirac spectrum.

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